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ASYMPTOTIC PROPERTIES OF THE HEAT KERNEL ON CONIC MANIFOLDS

BY

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ABSTRACT

We derive asymptotic properties for the heat kernel of elliptic cone (or Fuchs type) differential operators on compact manifolds with boundary. Applications include asymptotic formulas for the heat trace, counting function, spectral function, and zeta function of cone operators.

1. Introduction

We begin by discussing cone operators and their heat kernels. Let E be a Hermitian vector bundle over a compact manifold X with (connected) boundary $Y = \partial X$, on which there is a fixed boundary defining function x and a fixed b-measure $d\mathfrak{m}$. Here, a b-measure is a density of the form $x^{-1}\times$ a smooth positive density on X. A cone differential operator is an operator of the form $A = x^{-m}P$, where $P \in \mathrm{Diff}_b^m(X, E)$ is a "totally characteristic" (or b-) differential operator. Hence, A is a usual differential operator of order m on the interior of X such that in any collar decomposition $X \cong [0,\varepsilon)_x \times Y$ near Y over which $E \cong E|_Y$, A takes the form

$$A = x^{-m}\sum_{k=0}^{m} A_{m-k}(x)(xD_x)^k, \quad D_x = \frac{1}{i}\partial_x, \tag{1.1}$$

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where $A_{m-k}(x)$ is a differential operator of order $m-k$ on Y depending smoothly on x. The primary examples of cone differential operators are Dirac operators and Laplacians associated to a conic metric on X.

Under natural ellipticity conditions on $A \in x^{-m}\mathrm{Diff}_b^m(X,E)$, called "full" or "parameter" ellipticity, Gil [11] shows that the heat operator e^{-tA} exists as an operator between weighted Sobolev spaces. Thus, for some $\alpha \in \mathbb{R}$ we have

$$(1.2) \qquad e^{-tA}: x^{\alpha-m}L_b^2(X,E) \longrightarrow x^{\alpha}H_b^{\ell}(X,E), \quad \text{for any } \ell \in \mathbb{N}_0,$$

where $L_b^2(X,E)$ denotes the sections of E that are square integrable with respect to $d\mathfrak{m}$, and $H_b^{\ell}(X,E)$ consists of those $u \in L_b^2(X,E)$ such that $\mathrm{Diff}_b^{\ell}(X,E)u \subset L_b^2(X,E)$. Gil also attains the following trace expansion: As $t \to 0$,

$$(1.3) \qquad \mathrm{Tr}\, e^{-tA} \sim \sum_{k=0}^{\infty} a_k t^{(k-n)/m} + \sum_{k=0}^{\infty} b_k t^{k/m} \log t, \quad \text{where } n = \dim X.$$

Trace expansion of cone operators has a long history stemming from Cheeger's paper [6] on the cone Laplacian, and has proceeded through many developments in analysis on conic manifolds; see, for instance, Callias [5], Cheeger [7], Chou [9], Brüning–Seeley [3], Brüning–Lesch [2], Lesch [16], Koral' [14], and Mooers [24]. We note that Mooers achieves the expansion (1.3) for the cone Laplacian utilizing similar "blow-up" techniques as those featured in this paper.

For our first key result, we generalize (1.3) by adding a differential factor and give formulas for certain coefficients in the expansion.

THEOREM 1.1: *Let $B \in x^{-\beta}\mathrm{Diff}_b^{m'}(X,E)$, where $m' \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$ with $\beta < m$. Then Be^{-tA} is trace class on $x^{\alpha-m}L_b^2(X,E)$ for $t > 0$, and as $t \to 0$ we have*

$$(1.4) \qquad \mathrm{Tr}\, Be^{-tA} \sim \sum_{k=0}^{\infty} a_k t^{z_k} + \sum_{k=0}^{\infty} \{b_k \log t + c_k\} t^{(k-\beta)/m},$$

where $z_k = (k - m' - n)/m$. Moreover, if $\ell = z_k$, then

$$(1.5) \qquad a_k = \begin{cases} \frac{\Gamma(-\ell)}{m} {}^b\!\!\int_X \mathrm{Res}(BA^{\ell}), & \text{if } \ell \notin \mathbb{N}_0; \\ \frac{(-1)^{\ell+1}}{m\cdot \ell!} {}^b\!\!\int_X \mathrm{Res}(B \log A A^{\ell}), & \text{if } \ell \in \mathbb{N}_0. \end{cases}$$

Here, $\Gamma(z)$ is the Gamma function, $\mathrm{Res}(BA^{\ell})$ and $\mathrm{Res}(B\log AA^{\ell})$ are (Wodzicki) residue densities which are defined in (3.11) and are discussed in the latter part of Section 3.1, and finally, ${}^b\!\!\int_X$ denotes the regularized (or b-) integral over X

(see Section 3.2). If $\ell = (k-\beta)/m$, then

$$(1.6) \qquad b_k = \begin{cases} -\frac{\Gamma(-\ell)}{m^2 k!} \int_Y \partial_x^k \{x^k \operatorname{Res}(BA^\ell)\}|_{x=0}, & \text{if } \ell \notin \mathbb{N}_0; \\ \frac{(-1)^\ell}{m^2 \cdot k! \cdot \ell!} \int_Y \partial_x^k \{x^k \operatorname{Res}(B \log A A^\ell)\}|_{x=0}, & \text{if } \ell \in \mathbb{N}_0. \end{cases}$$

Lastly, if $\beta = 0$, then the constant term of (1.4) is

$$(1.7) \qquad -\frac{1}{m} {}^b\!\!\int_X \operatorname{Res}(B \log A) + \operatorname{Res}_0\{\Gamma(z)\hat{\zeta}_A(z;B)\}.$$

The function $\hat{\zeta}_A(z;B)$ is described below, and $\operatorname{Res}_0\{\Gamma(z)\hat{\zeta}_A(z;B)\}$ denotes the regular value of $\Gamma(z)\hat{\zeta}_A(z;B)$ at $z=0$.

If $X \cong [0,\varepsilon)_x \times Y_y$ near Y, then the diagonal in X^2 has the same decomposition near its boundary. Thus, if $\operatorname{tr} Be^{-tA}$ denotes the pointwise trace of Be^{-tA} on the diagonal, then near $x=0$, $\operatorname{tr} Be^{-tA}$ is a function of t, x, and y. Integrating out the y variable, we show that if $s = t^{1/m}$ and $v = x/s$, then $\int_Y \operatorname{tr} Be^{-tA} = k(s,v)$, where $k(s,v)$ is smooth for $s \in [0,\infty)$ and $v \in (0,\infty)$. Then,

$$\hat{\zeta}_A(z;B) := \frac{1}{\Gamma(z)} \int_0^\infty v^z k(0,v) \frac{dv}{v},$$

and we show that $\hat{\zeta}_A(z;B)$ is a meromorphic function on $\mathbb{C}$. The function $\hat{\zeta}_A(z;B)$ for $B = \mathrm{Id}$ was first studied by Lesch in [16, Sec. 2.2] where it was used to define the eta invariant of a cone operator, which appears in an index theorem, see [16, Cor. 2.4.7]. Related index theorems can be found in [9, 7, 4].

Our second main result (see Theorem 3.1) describes the Schwartz kernel of e^{-tA} as a polyhomogeneous function on a blown-up manifold. Actually, to simplify exposition, we don't define the blown-up manifold, but rather we describe the kernel of e^{-tA} using coordinates. Understanding the heat kernel on a blown-up manifold was initiated by Melrose [22], and was developed by Mooers [24] for the cone Laplacian. The precise description of the heat kernel can be used to extract analytic properties of the kernel of the complex powers A^z of A; see [18].

We now discuss various applications. Our first application is concerned with the zeta function of A. In [18] it is proved that, under certain conditions on the resolvent $(A-\lambda)^{-1}$, the complex power A^z exists as an entire family of b-pseudodifferential operators. Using the expansion (1.4) we prove the following.

THEOREM 1.2 (Zeta Function): *$z \mapsto \operatorname{Tr} BA^z$ is defined and holomorphic for $\operatorname{Re} z < \min\{(-m'-n)/m, -\beta/m\}$; and extends to be meromorphic on the whole* *complex plane, with (possible) simple poles on the set* $\{z_k, (k-\beta)/m \mid k \in \mathbb{N}_0\}$, *and with (possible) double poles on the set* $\{(k-\beta)/m \mid k \in \mathbb{N}_0, (k-\beta)/m \notin \mathbb{N}_0\}$.

Assume that A: $x^\alpha H_b^m(X,E) \longrightarrow x^{\alpha-m} L_b^2(X,E)$ is self-adjoint and positive (the number α is the same one that appears in (1.2)). We describe a couple more applications that deal with the counting function $N(\lambda)$, the number of eigenvalues of A less than $\lambda \in \mathbb{R}$, and the spectral function of A:

$$e(p,q,\lambda) = \sum_{\lambda_j < \lambda} e_j(p)\langle \,\cdot\, , e_j(q)\rangle, \quad p,q \in X, \quad Ae_j = \lambda_j e_j,$$

where the sum includes multiplicity of the eigenvalue, and where the e_j's are orthonormal. (That A has discrete spectrum follows from [16, Prop. 1.4.7].)

Let $a(p,\xi)$ be the principal symbol of A, and define

$$c_A = \frac{1}{n(2\pi)^n} \int_{S^*X} \operatorname{tr}\{a(p,\omega)^{-n/m}\} dp d\omega,$$

where $S^*X = (T^*X \setminus 0)/\mathbb{R}^+$ is the cosphere bundle, and the form $dpd\omega$ is defined by contracting the n-th power of the canonical symplectic form on T^*X with the radial vector field. Then the trace expansion (1.4) implies the following.

THEOREM 1.3 (Weyl Asymptotics): *As* $\lambda \to \infty$, *we have*

$$N(\lambda) - c_A \lambda^{n/m} = o(\lambda^{n/m}). \tag{1.8}$$

The right-hand side can be replaced with $\mathcal{O}(\lambda^{(n-1)/m})$ for the scalar Laplacian on a conic manifold; see Kalka–Ménikoff [13] and Phạm The Lại–Petkov [15]. The estimate (1.8) in the generality considered here also follows from Karol' [14].

Let $a_b(p,\xi)$ be the totally characteristic (or b-) principal symbol of $x^m A$. Here, $(p,\xi) \in {}^bT_p^*X$, the b-cotangent bundle, with ξ the fiber variable. Let

$$c_A(p) = \frac{1}{n(2\pi)^n} \int_{{}^bS_p^*X} \operatorname{tr}\{a_b(p,\omega)^{-n/m}\} d_b\omega,$$

where ${}^bS^*X = ({}^bT^*X \setminus 0)/\mathbb{R}^+$ is the b-cosphere bundle, and where the density $d_b\omega$ is defined by contracting the n-th power of the canonical symplectic form on ${}^bT_p^*X$ with the radial vector field and then dividing this form by $d\mathfrak{m}(p)$, the fixed b-measure on X. Our description of the heat kernel e^{-tA} implies the following.

THEOREM 1.4 (Asymptotics of the Spectral Function): *As* $\lambda \to \infty$, *we have*

$$\operatorname{tr} e(p,p,\lambda/x^m) - c_A(p)\lambda^{n/m} = o(\lambda^{n/m}) \tag{1.9}$$

where $x = x(p)$ is the boundary defining function evaluated at $p \in X$, and where the o estimate is uniform for $p \in X$, including up to the boundary.

The spectral function (or its Fourier transform) for the scalar cone Laplacian has been investigated by Cheeger–Taylor [8], Kalka–Ménikoff [13], Phạm The Lại–Petkov [15], and Melrose–Wunsch [23], among others.

In Section 2, we review a space of parameter-dependent operators that will be used to understand the structure of the resolvent of an elliptic cone operator. In Section 3, we use the resolvent structure to analyze the heat operator. In particular, in Section 3.1 we describe the polyhomogeneous nature of the heat kernel and in Section 3.2 we prove Theorem 1.1. Finally, in Section 4 we prove Theorem 1.2, Theorem 1.3, and Theorem 1.4.

In conclusion, I thank the referee for helpful comments in improving this paper.

2. The resolvent of cone differential operators

The material in this section is taken from [19] and [20]. To simplify the exposition, we will henceforth assume that $E = \mathbb{C}$ is the trivial bundle. We make this simplification so that definitions and theorems are less cumbersome to state. However, there are analogous statements when vector bundles are present.

We begin by describing full-ellipticity. We use the same notation as in the introduction. Let $A \in x^{-m}\mathrm{Diff}^m_b(X)$ be a cone differential operator. If A is written in the form (1.1) near Y, then we associate to A the operator

$$I(A) = \rho^{-m} \sum_{k=0}^{m} A_{m-k}(0)(\rho D_\rho)^k.$$

We denote by ${}^b\sigma_m(x^m A)(\xi)$ the totally characteristic (or b-) principal symbol of $x^m A$; see [22, Sec. 2.4]. The boundary spectrum, $\mathrm{spec}_c(A) \subset \mathbb{C}$, consists of points $\tau \in \mathbb{C}$ where the holomorphic family

$$\tau \mapsto \sum_{k=0}^{m} A_{m-k}(0)\tau^k \colon H^m(Y) \longrightarrow L^2(Y)$$

fails to be invertible; see [22, Sec. 5.1]. On the manifold $Y^\wedge = [0,\infty)_\rho \times Y$ we define the spaces $H^{\ell,\alpha}_c(Y^\wedge)$, $\ell \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$ as follows. Let $\chi \in C^\infty_c([0,\infty))$ with $\chi(\rho) = 1$ near $\rho = 0$. Then $H^{\ell,\alpha}_c(Y^\wedge)$ consists of distributions u on $Y^\wedge$ such that $\chi u \in \rho^\alpha H^\ell_b(Y^\wedge)$ and such that, given any coordinate patch $\mathcal{U}$ on Y diffeomorphic to an open subset of $\mathbb{S}^{n-1}$ and function $\varphi \in C^\infty_c(\mathcal{U})$, we have $(1-\chi)\varphi u \in H^\ell(\mathbb{R}^n)$ where $(0,\infty) \times \mathbb{S}^{n-1}$ is identified with $\mathbb{R}^n \setminus \{0\}$ via polar coordinates.

Definition 2.1: Let $\Lambda \subset \mathbb{C}$ be a closed sector (a closed angle with vertex at 0). Then $A \in x^{-m}\mathrm{Diff}_b^m(X)$ is **fully elliptic with respect to $\alpha \in \mathbb{R}$ on Λ** if

(1) ${}^b\sigma_m(x^m A)(\xi) - \lambda$ is invertible for all $\xi \neq 0$ and $\lambda \in \Lambda$.

(2) $\alpha \notin -\mathrm{Im}\,\mathrm{spec}_c(A)$.

(3) $I(A) - \lambda\colon H_c^{m,\alpha}(Y^\wedge) \longrightarrow H_c^{0,\alpha-m}(Y^\wedge)$ is invertible for all $\lambda \in \Lambda$ sufficiently large.

Remark 2.2: "Full-ellipticity" is the terminology of Melrose; in the terminology of Gil [11], full-ellipticity is called "parameter-ellipticity".

If A is fully elliptic with respect to α on a sector Λ, then Gil [11] proves that

$$A - \lambda\colon x^\alpha H_b^m(X) \longrightarrow x^{\alpha-m} L_b^2(X)$$

is invertible for $\lambda \in \Lambda$ sufficiently large. Our goal is to obtain precise information on the heat operator

$$e^{-tA} = \frac{i}{2\pi}\int e^{-t\lambda}(A-\lambda)^{-1}d\lambda$$

by first obtaining precise information on the kernel of the resolvent. A space of parameter-dependent operators that can be used to extract precise information on the resolvent was developed in [19]. To explain this program, we start by describing their corresponding symbols. Similar symbols can be found in, e.g., Grubb–Seeley [12] and Shubin [25].

Given $m \in \mathbb{R}$ and $d \in \mathbb{Z}^+$, we denote by $S_\Lambda^{m,d}(\mathbb{R}^n)$ the space of functions $a \in C^\infty(\Lambda \times \mathbb{R}^n)$ satisfying the following estimates: for each α, β,

$$|\partial_\lambda^\alpha \partial_\xi^\beta a(\lambda,\xi)| \le C(1 + |\lambda|^{1/d} + |\xi|)^{m-d|\alpha|-|\beta|}.$$

The corresponding classical subspace is defined as follows: Given $m \in \mathbb{R}$ and $d \in \mathbb{Z}^+$, the space $S_{\Lambda,c\ell}^{m,d}(\mathbb{R}^n)$ consists of those $a(\lambda,\xi) \in S_\Lambda^{m,d}(\mathbb{R}^n)$ such that

$$a(\lambda,\xi) \sim \sum_{j=0}^{\infty} \chi(\lambda,\xi) a_{m-j}(\lambda,\xi), \tag{2.1}$$

where $\chi(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ with $\chi(\lambda,\xi) = 0$ near $(\lambda,\xi) = 0$ and $\chi(\lambda,\xi) = 1$ outside a neighborhood of 0, where $a_{m-j}(\lambda,\xi)$ is a smooth function of $(\lambda,\xi) \in \Lambda \times \mathbb{R}^n \setminus \{(0,0)\}$ such that $a_{m-j}(\delta^d\lambda, \delta\xi) = \delta^{m-j} a_{m-j}(\lambda,\xi)$ for all $\delta > 0$, and finally, where the asymptotic sum (2.1) means that for each $N \in \mathbb{Z}^+$,

$$a(\lambda,\xi) - \sum_{j=0}^{N-1} \chi(\lambda,\xi) a_{m-j}(\lambda,\xi) \in S_\Lambda^{m-N,d}(\mathbb{R}^n).$$

The symbol $a(\lambda, \xi) \in S^{m,d}_{\Lambda,c\ell}(\mathbb{R}^n)$ is said to be holomorphically tempered if it is holomorphic on a neighborhood of Λ and if there exists an $\varepsilon > 0$ such that each homogeneous component $a_{m-j}(\lambda, \xi)$ extends to be a smooth function, holomorphic in λ, for (λ, ξ) in the region

$$\{(\lambda, \xi) \in \mathbb{C} \times (\mathbb{R}^n \setminus \{0\}) |\ \lambda \in \Lambda \quad \text{or} \quad |\lambda| \le \varepsilon |\xi|^d \quad \text{or} \quad \frac{1}{\varepsilon}|\xi|^d \le |\lambda|\}.$$

We are now ready to define our spaces of parameter-dependent cone operators. The Schwartz kernels of these operators are associated with the blown-up manifold X^2_b, which is "X^2 blown-up along $Y \times Y$". The now familiar picture of X^2_b, along with its various submanifolds, is shown in Figure 1. For more on blow-ups see [10] or [21]. Let $d\mathfrak{m}'$ denote the b-density $d\mathfrak{m}$ lifted to X^2 under the right projection $X^2 \ni (p, q) \mapsto q \in X$, and fix a boundary defining function ϱ for ff.

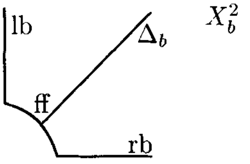

Figure 1. A geometric picture of X^2_b. The submanifold Δ_b is the diagonal of X^2 lifted to X^2_b.

Let $m \in \mathbb{R}$ and $d \in \mathbb{Z}^+$. Then we define $\Psi^{m,d}_{c,\Lambda}(X)$ as the space of operator families $Q(\lambda)$ defined for $\lambda \in \Lambda$ that have a Schwartz kernel $K_{Q(\lambda)}$ satisfying the following two conditions:

(1) Given $\varphi \in C^\infty_c(X^2_b \setminus \Delta_b)$, the kernel $\varphi K_{Q(\lambda)}$ is of the form $k(\varrho^d \lambda, p) d\mathfrak{m}'$, where $k(\lambda, p)$ is a smooth function of $(\lambda, p) \in \Lambda \times X^2_b$, and where $k(\lambda, p)$ vanishes to infinite order (that is, with all derivatives) at the sets $\Lambda \times \text{lb}$ and $\Lambda \times \text{rb}$ and as $|\lambda| \to \infty$ in Λ.

(2) Given a coordinate patch of X^2_b overlapping Δ_b of the form $\mathcal{U}_y \times \mathbb{R}^n_z$ such that $\Delta_b \cong \mathcal{U} \times \{0\}$ and given $\varphi \in C^\infty_c(\mathcal{U} \times \mathbb{R}^n)$, we have

$$\varphi K_{Q(\lambda)} = \int e^{iz\cdot\xi} q(\varrho^d \lambda, y, \xi) đ\xi \cdot d\mathfrak{m}', \quad đ\xi = \frac{1}{(2\pi)^n} d\xi,$$

where $y \mapsto q(\lambda, y, \xi) \in C^\infty(\mathcal{U}; S^{m,d}_{\Lambda,c\ell}(\mathbb{R}^n))$.

If, in addition, $Q(\lambda)$ is holomorphic on a neighborhood of Λ, and if its local symbols take values in the holomorphically tempered symbols, then $Q(\lambda)$ is called holomorphically tempered.

Remark 2.3: The space of b-pseudodifferential operators $\Psi_b^m(X)$ is defined in the same way except that in (1), the kernel is of the form $k(p)d\mathfrak{m}'$, where $k(p)$ is a smooth function on X_b^2 that vanishes to infinite order at lb and rb; and in (2), the symbol is of the form $q(y,\xi)$, where $y \mapsto q(y,\xi) \in C^\infty(\mathcal{U}; S_{c\ell}^m(\mathbb{R}^n))$ with $S_{c\ell}^m(\mathbb{R}^n)$ denoting the classical symbols on $\mathbb{R}^n$ of order m.

Before we introduce our next spaces of parameter-dependent operators, we review asymptotic expansions. An index set is a subset $F \subset \mathbb{C} \times \mathbb{N}_0$ such that if $(z,k) \in F$, then $(z+\ell, j) \in F$ for all $\ell \in \mathbb{N}_0$ and $0 \le j \le k$, and such that given any $N \in \mathbb{N}_0$, $\{(z,k) \in F | \operatorname{Re} z \le N\}$ is a finite set. Then a smooth function u on the interior of X has an asymptotic expansion at Y with index set F if it has the property that given any $N \in \mathbb{Z}^+$, on a collar $[0,\varepsilon)_x \times Y_y$ we have

$$\left| u(x,y) - \sum_{(z,k)\in F, \operatorname{Re} z \le N} x^z (\log x)^k u_{(z,k)}(y) \right| \le C x^N, \tag{2.2}$$

for some functions $u_{(z,k)}(y)$ on Y. Given any b-differential operator P, we require that Pu has the same property. For any manifold with corners M, one can define an asymptotic expansion at a boundary hypersurface H of M with index set F in a similar fashion; see [22, Sec. 5.10] or the appendices of [10] or [21]. Essentially, one requires a condition similar to (2.2) to hold in any collar of H.

Recall that $d\mathfrak{m}'$ denotes the b-density $d\mathfrak{m}$ lifted to X^2 under the right projection $X^2 \ni (p,q) \mapsto q \in X$. Let $\mathcal{F} = (F_{\mathrm{lb}}, F_{\mathrm{rb}}, F_{\mathrm{ff}}, F)$ be a set of four index sets. We define $\Psi_{c,\Lambda}^{-\infty,d,\mathcal{F}}(X)$ as the class of operator families $R(\lambda)$ depending smoothly on $\lambda \in \Lambda$ that have a Schwartz kernel $K_{R(\lambda)}$ satisfying the following two conditions:

1. Given $\varphi \in C_c^\infty(X_b^2 \setminus \mathrm{ff})$, the kernel $\varphi K_{R(\lambda)}$ is of the form $k(\lambda,p)d\mathfrak{m}'$, where $k(\lambda,p)$ is a smooth function of $(\lambda,p) \in \Lambda \times \operatorname{int}(X_b^2)$ that vanishes to infinite order as $|\lambda| \to \infty$ and can be expanded at the sets $\Lambda \times \mathrm{lb}$ and $\Lambda \times \mathrm{rb}$ with index sets F_{lb} and F_{rb} respectively.
2. Let $[0,\varepsilon)_\varrho \times \mathrm{ff}_y$ be a collar of ff in X_b^2 and let $\varphi \in C_c^\infty([0,\varepsilon)_\varrho \times \mathrm{ff}_y)$. Then for λ in compact subsets of Λ, the kernel $\varphi K_{R(\lambda)}$ can be expanded at $\varrho = 0$, $y \in \mathrm{lb}$, and $y \in \mathrm{rb}$ with index sets F_{ff}, F_{lb}, and F_{rb} respectively. For λ large, $\varphi K_{R(\lambda)}$ can be written in the form $k(r,v,\theta,y)d\mathfrak{m}'$, where $r = |\lambda|^{-1/d}$, $v = \varrho|\lambda|^{1/d}$, and $\theta = \lambda/|\lambda|$. Moreover, k vanishes to infinite order as $v \to \infty$; is smooth in θ; and k has expansions at $r=0$, $v=0$, $y \in \mathrm{lb}$, and $y \in \mathrm{rb}$ with index sets F, F_{ff}, F_{lb}, and F_{rb} respectively.

Our third and final space of operators is defined as follows. Let $\mathcal{G} = (G_{\mathrm{lb}}, G_{\mathrm{rb}})$ be a pair of index sets. Here, lb represents the left boundary $Y \times X$ of X^2 and rb the right boundary $X \times Y$ of X^2. The space $\Psi_\Lambda^{-\infty,\mathcal{G}}(X)$ consists of integral operators $S(\lambda)$ that have a Schwartz kernel of the form $k(\lambda, p)d\mathfrak{m}'$, where $k(\lambda, p)$ is a function on $\Lambda \times X^2$ that is smooth in $\lambda \in \Lambda$, vanishing to infinite order as $|\lambda| \to \infty$ in Λ, and can be expanded at the sets $\Lambda \times \mathrm{lb}$ and $\Lambda \times \mathrm{rb}$ with index sets G_{lb} and G_{rb} respectively.

Given any $s \in \mathbb{R}$, $H_b^s(X)$ consists of those distributions u such that $\Psi_b^s(X)u \subset L_b^2(X)$. Then (see e.g. [22]) any $A \in x^{-m}\mathrm{Diff}_b^m(X)$ defines a continuous linear map

$$A\colon x^\alpha H_b^s(X) \longrightarrow x^{\alpha-m}H_b^{s-m}(X) \quad \text{for any } \alpha, s \in \mathbb{R}. \tag{2.3}$$

Our main result concerning resolvents is the following.

THEOREM 2.4 ([19, Th. 6.1]): *Let $A \in x^{-m}\mathrm{Diff}_b^m(X)$, $m \in \mathbb{Z}^+$, be fully elliptic with respect to $\alpha \in \mathbb{R}$ on a sector Λ. Then for $\lambda \in \Lambda$ sufficiently large,*

$$A - \lambda\colon x^\alpha H_b^s(X) \longrightarrow x^{\alpha-m}H_b^{s-m}(X) \quad \text{for any } s \in \mathbb{R},$$

is invertible, and given $B \in x^{-\beta}\mathrm{Diff}_b^{m'}(X)$ where $\beta \in \mathbb{R}$ and $m' \in \mathbb{N}_0$, we have

$$B(A-\lambda)^{-1} = Q(\lambda) + R(\lambda) + S(\lambda),$$

where $Q(\lambda) \in x^{m-\beta}\Psi_{c,\Lambda}^{m'-m,m}(X)$ is holomorphically tempered, and where $R(\lambda) \in x^{-\beta}\Psi_{c,\Lambda}^{-\infty,m,\mathcal{F}(\alpha)}(X)$ and $S(\lambda) \in x^{-\beta}\Psi_{\Lambda}^{-\infty,\mathcal{G}(\alpha)}(X)$ for some index families $\mathcal{F}(\alpha)$ and $\mathcal{G}(\alpha)$.

As noted, this result is just [19, Th. 6.1]. Actually, the theorem of loc. cit. was established without the factor of B or the holomorphically tempered condition, but the proof can be easily modified to accommodate these extra features. The index sets $\mathcal{F}(\alpha)$ and $\mathcal{G}(\alpha)$ are defined as follows; cf. [19, Sec. 3]. The order of a pole $\tau \in \mathrm{spec}_c(A)$ is denoted by $\mathrm{ord}(\tau)$. We define

$$\begin{aligned}\widehat{E}^{\pm}(\alpha) = \{(z+r,k) \mid\, & r \in \mathbb{N}_0, \tau = \mp iz \in \mathrm{spec}_c(A) + im,\\ & 1 \le k+1 \le \sum_{\ell=0}^{r} \mathrm{ord}(\tau - im \mp i\ell), \text{ and } \mathrm{Re} z > \pm(\alpha - m)\}.\end{aligned}$$

For index sets E and F, we define $E\overline{\cup}F = E \cup F \cup \{(z, k+\ell+1) \mid (z,k) \in E, (z,\ell) \in F\}$. Set $\check{E}^{\pm}(\alpha) = \widehat{E}^{\pm}(\alpha)\overline{\cup}\widehat{E}^{\pm}(\alpha)$. Then,

$$\mathcal{G}(\alpha) = (G_{\mathrm{lb}}(\alpha), G_{\mathrm{rb}}(\alpha)) = (\check{E}^{+}(\alpha) + m, \check{E}^{-}(\alpha)), \tag{2.4}$$

$$\mathcal{F}(\alpha) = (\mathcal{G}(\alpha), E(\alpha), \mathbb{N}_0 + m), \quad E(\alpha) = \mathbb{Z}^+\overline{\cup}(\widehat{E}^{+}(\alpha) + \widehat{E}^{-}(\alpha)) + m. \tag{2.5}$$

Later we will need the following index set:

$$\widetilde{E}(\alpha) = (\check{E}^{+}(\alpha) + \check{E}^{-}(\alpha) + m) \cup E(\alpha). \tag{2.6}$$

3. The heat operator

Henceforth, we assume that our sector Λ is of the form

(3.1) Assumption: $\Lambda = \{\lambda \in \mathbb{C} |\ \varepsilon_0 \le \arg(\lambda) \le 2\pi - \varepsilon_0\}$, where $0 < \varepsilon_0 < \pi/2$,

and we let Γ denote an anti-clockwise contour in Λ of the form

(3.2) $\Gamma = a + \{\lambda \in \mathbb{C} |\ \arg(\lambda) = \delta \text{ or } \arg(\lambda) = 2\pi - \delta\}, \quad a < 0, \quad \varepsilon_0 < \delta < \pi/2.$

Let $A \in x^{-m}\mathrm{Diff}_b^m(X)$, $m \in \mathbb{Z}^+$, be fully elliptic with respect to $\alpha \in \mathbb{R}$ on Λ. Then by Theorem 2.4, $(A-\lambda)^{-1}$ exists for $\lambda \in \Lambda$ sufficiently large. Let Γ be any contour in Λ of the form given in (3.2) such that $(A-\lambda)^{-1}$ exists on and outside of Γ. Then the heat operator of A is defined by

$$e^{-tA} = \frac{i}{2\pi} \int_\Gamma e^{-t\lambda}(A-\lambda)^{-1} d\lambda, \quad t > 0.$$

3.1. The Schwartz kernel. Let $B \in x^{-\beta}\mathrm{Diff}_b^{m'}(X)$, $\beta \in \mathbb{R}$, $m' \in \mathbb{N}_0$. Our goal is to describe the Schwartz kernel of Be^{-tA}. To do so, we use Theorem 2.4 to write $B(A-\lambda)^{-1} = Q(\lambda) + R(\lambda) + S(\lambda)$. Hence,

$$Be^{-tA} = Q(t) + T(t),$$

where

$$(3.3) \qquad Q(t) = \frac{i}{2\pi}\int_\Gamma e^{-t\lambda} Q(\lambda) d\lambda, \quad T(t) = \frac{i}{2\pi}\int_\Gamma e^{-t\lambda}(R(\lambda) + S(\lambda)) d\lambda.$$

Theorem 3.1: *The following properties hold:*

(A) *Let $\varphi \in C_c^\infty(X_b^2 \setminus \Delta_b)$. Then, $\varphi K_{Q(t)}$ is of the form $x^{-\beta} k_Q(t/\varrho^m, p) d\mathfrak{m}'$, where $k_Q(t,p)$ is a smooth function of $(t,p) \in [0,\infty) \times X_b^2$ vanishing to infinite order at $t = 0$, $t \to \infty$, $p \in \mathrm{lb}$, and $p \in \mathrm{rb}$.*

(B) *Let $\varphi \in C^\infty(X_b^2)$ have support in a coordinate patch $\mathcal{U}_y \times \mathbb{R}_z^n$ of X_b^2 overlapping Δ_b such that $\Delta_b \cong \mathcal{U} \times \{0\}$. Then we can write*

$$\varphi K_{Q(t)} = x^{-\beta} \int e^{iz\cdot\xi} q\Big(\frac{t}{\varrho^m}, y, \xi\Big) đ\xi \cdot d\mathfrak{m}',$$

where $q \in C^\infty([0,\infty)_t \times \mathcal{U}_y \times \mathbb{R}_\xi^n)$ and has the following properties:

(a) *There exists a constant $\eta > 0$ such that $q(t,y,\xi)$ satisfies the estimates: for any k, γ, and σ,*

$$(3.4) \qquad |\partial_t^k \partial_y^\gamma \partial_\xi^\sigma q(t,y,\xi)| \le C(1+|\xi|)^{m'+mk-|\sigma|} e^{-t\eta(1+|\xi|^m)}.$$

(b) *If $N \in \mathbb{Z}^+$, then we can write*

$$q(t,y,\xi) = \sum_{j=0}^{N-1} q_j(t,y,\xi) + r_N(t,y,\xi),$$

where $q_j, r_N \in C^\infty([0,\infty)_t \times \mathcal{U}_y \times \mathbb{R}^n_\xi)$, and satisfy

(i) *$q_j(\delta^{-m}t, y, \delta\xi) = \delta^{m'-j} q_j(t,y,\xi)$ for all $\delta > 0$;*

(ii) *$q_j(t,y,\xi)$ satisfies the estimate (3.4) with mk replaced with $mk - j$ and with $e^{-t\eta(1+|\xi|^m)}$ replaced with $e^{-t\eta|\xi|^m}$;*

(iii) *r_N satisfies the estimate (3.4) with mk replaced with $mk - N$ and with $e^{-t\eta(1+|\xi|^m)}$ replaced with $e^{-t\eta|\xi|^m}$.*

(C) *$Q(t) \to B$ continuously in $x^{-\beta}\Psi^{m'}_b(X)$ as $t \to 0$.*

(D) *If $t = s^m$, then the kernel of $T(t)$ is of the form*

$$K_{T(t)} = x^{-\beta} k_T\Big(s, \frac{\varrho}{s}, p\Big) d\mathfrak{m}', \quad p \in X^2_b,$$

where $k_T(s,v,p)$ is smooth in $p \in X^2_b$ except at lb *and* rb *where it has expansions with index sets $G_{\mathrm{lb}}(\alpha)$ and $G_{\mathrm{rb}}(\alpha)$ respectively; smooth in $s \in [0,\infty)$; and smooth in $v \in (0,\infty)$, vanishes to infinite order as $v \to \infty$, and can be expanded at $v = 0$ with the index set $\widetilde{E}(\alpha)$. Here, the index sets $G_{\mathrm{lb}}(\alpha)$, $G_{\mathrm{rb}}(\alpha)$, and $\widetilde{E}(\alpha)$ appear in (2.4) and (2.6).*

Proof: We begin with (A). Since $Q(\lambda) \in x^{m-\beta}\Psi^{m'-m,m}_{c,\Lambda}(X)$, by definition of this space it follows that we can write $\varphi K_{Q(\lambda)}$ in the form $x^{-\beta}\varrho^m \tilde{k}_Q(\varrho^m\lambda, p) d\mathfrak{m}'$, where $\tilde{k}_Q(\lambda, p)$ is a smooth function of $(\lambda, p) \in \Lambda \times X^2_b$ vanishing to infinite order at the sets $\Lambda \times \mathrm{lb}$ and $\Lambda \times \mathrm{rb}$ and as $|\lambda| \to \infty$ in Λ. It follows that

$$x^\beta \varphi K_{Q(t)} = \frac{i}{2\pi} \int_\Gamma e^{-t\lambda} \varrho^m \tilde{k}_Q(\varrho^m\lambda, p) d\lambda \cdot d\mathfrak{m}'.$$

Making the change of variables $\lambda \mapsto \varrho^{-m}\lambda$ and using the fact that $Q(\lambda)$ is holomorphic so that the contour $\varrho^m\Gamma$ can be shifted back to Γ, we can write

$$x^\beta \varphi K_{Q(t)} = \frac{i}{2\pi} \int_\Gamma e^{-(t/\varrho^m)\lambda} \tilde{k}_Q(\lambda, p) d\lambda \cdot d\mathfrak{m}' = k_Q(t/\varrho^m, p) d\mathfrak{m}',$$

where

$$k_Q(t,p) = \frac{i}{2\pi} \int_\Gamma e^{-t\lambda} \tilde{k}_Q(\lambda, p) d\lambda.$$

Since $Q(\lambda)$ is holomorphic on a neighborhood of Λ, the contour Γ can be shifted to the right of the imaginary axis. It follows that $k_Q(t,p)$ vanishes exponentially as $t \to \infty$. The other asymptotic properties of $k_Q(t,p)$ follow directly from the properties of $\tilde{k}_Q(\lambda, p)$.

We now consider (B). Here we can write

$$x^\beta \varphi K_{Q(\lambda)} = \int e^{iz\cdot\xi} \tilde{q}(\varrho^m \lambda, y, \xi) đ\xi \cdot d\mathfrak{m}',$$

where $y \mapsto \tilde{q}(\lambda, y, \xi) \in C^\infty(\mathcal{U}; S^{m'-m,m}_{\Lambda,c\ell}(\mathbb{R}^n))$. Following the same line of reasoning used in the proof of Part (A) gives

$$x^\beta \varphi K_{Q(t)} = \int e^{iz\cdot\xi} q(t/\varrho^m, y, \xi) đ\xi \cdot d\mathfrak{m}',$$

where

$$q(t, y, \xi) = \frac{i}{2\pi} \int_\Gamma e^{-t\lambda} \tilde{q}(\lambda, y, \xi) d\lambda. \tag{3.5}$$

Now all the properties of $q(t,y,\xi)$ follow from [20, Sec. 5, 6], where we analyzed Laplace transforms such as (3.5). Moreover, the analysis in loc. cit. together with Part (A) prove Part (C).

Thus, we are left to prove (D). We focus on describing the Schwartz kernel of $T(t)$ near ff. Near ff, let $X^2_b \cong [0,\varepsilon)_\varrho \times \mathrm{ff}_y$. Now recall that $x^\beta R(\lambda) + x^\beta S(\lambda) \in \Psi^{-\infty,m,\mathcal{F}(\alpha)}_{c,\Lambda}(X) + \Psi^{-\infty,\mathcal{G}(\alpha)}_{\Lambda}(X) \subset \Psi^{-\infty,m,\widetilde{\mathcal{F}}(\alpha)}_{c,\Lambda}(X)$ where $\widetilde{\mathcal{F}}(\alpha) = (\mathcal{G}(\alpha), \widetilde{E}(\alpha), \mathbb{N}_0 + m)$ (this inclusion is straightforward to verify). Thus, we can write

$$x^\beta K_{R(\lambda)+S(\lambda)} = k(r, v, \theta, y) d\mathfrak{m}', \quad r = |\lambda|^{-1/m}, \quad v = \varrho|\lambda|^{1/m},$$

where $k(r,v,\theta,y)$ has expansions at $r=0$ with index set $\mathbb{N}_0 + m$; at $v = 0$ with index set $\widetilde{E}(\alpha)$; at $y \in \mathrm{lb}$ with index set $G_{\mathrm{lb}}(\alpha)$; at $y \in \mathrm{rb}$ with index set $G_{\mathrm{rb}}(\alpha)$; and vanishes to infinite order as $v \to \infty$. Now by (3.3), with $t = s^m$, we have

$$x^\beta K_{T(t)} = \frac{i}{2\pi} \int_\Gamma e^{-s^m\lambda} k(|\lambda|^{-1/m}, \varrho|\lambda|^{1/m}, \theta, y) d\lambda \cdot d\mathfrak{m}'.$$

Thus, we are left to show that

$$k_T(s, v, y) := \frac{i}{2\pi} \int_\Gamma e^{-s^m\lambda} k(|\lambda|^{-1/m}, vs|\lambda|^{1/m}, \theta, y) d\lambda$$

satisfies the properties listed in (D). To see this, observe that making the change of variables $\lambda \mapsto s^{-m}\lambda$ and using the fact that $R(\lambda) + S(\lambda)$ is holomorphic so that the contour $s^m\Gamma$ can be shifted back to Γ, we can write

$$k_T(s, v, y) = s^{-m} \frac{i}{2\pi} \int_\Gamma e^{-\lambda} k(s|\lambda|^{-1/m}, v|\lambda|^{1/m}, \theta, y) d\lambda.$$

In view of the asymptotic properties of $k(r, v, \theta, y)$, utilizing this integral expression for $k_T(s, v, y)$ it is straightforward to verify that $k_T(s, v, y)$ has the asymptotic properties listed in (D). Our proof is now complete. ∎

This theorem together with the mapping properties of b-pseudodifferential operators give two immediate corollaries. Given Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Recall that $H_b^s(X)$ is defined as the space of distributions u such that $\Psi_b^s(X)u \subset L_b^2(X)$.

COROLLARY 3.2: *Let $A \in x^{-m}\mathrm{Diff}_b^m(X)$, $m \in \mathbb{Z}^+$, be fully elliptic with respect to $\alpha \in \mathbb{R}$ on a sector Λ of the form (3.1). Then for any $s, s' \in \mathbb{R}$,*

$$e^{-tA} \in C^\infty((0,\infty)_t; \mathcal{B}(x^{\alpha-m}H_b^s(X), x^\alpha H_b^{s'}(X))), \tag{3.6}$$

and for each $k \in \mathbb{N}_0$ and $s \in \mathbb{R}$, down to $t = 0$ we have

$$e^{-tA} \in C^k([0,\infty)_t; \mathcal{B}(x^{\alpha-m}H_b^s(X), x^{\alpha-m-mk}H_b^{s-mk}(X))). \tag{3.7}$$

Moreover, $e^{-tA}|_{t=0} = \mathrm{Id}$ on $x^{\alpha-m}H_b^s(X)$, and e^{-tA} satisfies the heat equation

$$(\partial_t + A)e^{-tA} = 0, \quad t > 0. \tag{3.8}$$

Proof: The structure of the heat kernel in Theorem 3.1 along with the mapping properties of the "full" calculus of b-pseudodifferential operators (see [21, Th. 3.25] or [22, Ch. 5]) imply that (3.6) holds. The standard argument shows that the heat equation (3.8) is satisfied.

It remains to prove (3.7). Since $\partial_t^k e^{-tA} = (-A)^k e^{-tA}$ by (3.8), it suffices to prove (3.7) for $k = 0$ by the mapping properties of A in (2.3). But the $k = 0$ case follows again from Theorem 3.1 and the mapping properties of b-pseudodifferential operators [21], [22]. ∎

The following corollary might be interesting for those readers familiar with "blow-up". As always, let $t = s^m$. Let X_H denote the blown-up space $[[0,\infty)_s \times X; \{s = 0\} \times Y]$. We refer the reader to [10] or [21] for the definition of blow-up. Let β: $X_H \longrightarrow [0,\infty)_s \times X$ be the blow-down map and set $\mathrm{bx} = \beta^*(Y)$; $\mathrm{tf} = \beta^*(\{s = 0\} \times Y)$; and $\mathrm{tb} = \beta^*(\{s = 0\} \times X)$. For the definition of the polyhomogeneous spaces $\mathcal{A}_{phg}$ appearing the following result, see [21].

COROLLARY 3.3: *Let $A \in x^{-m}\mathrm{Diff}_b^m(X)$, $m \in \mathbb{Z}^+$, be fully elliptic with respect to $\alpha \in \mathbb{R}$ on a sector Λ of the form (3.1). Then given any index set F with $\mathrm{Re}F > \alpha - m$, the heat operator defines a continuous linear map*

$$e^{-tA}\colon \mathcal{A}_{phg}^F(X) \longrightarrow \mathcal{A}_{phg}^{\mathcal{G}}(X_H),$$

where $\mathcal{G} = (G_{\rm bx}, G_{\rm tf}, G_{\rm tb}) = (G_{\rm lb}(\alpha)\overline{\cup}(E(\alpha)+F), F, \mathbb{N}_0)$. *Here, the index sets* $G_{\rm lb}(\alpha)$ *and* $E(\alpha)$ *appear in (2.5).*

Proof: This result follows directly from the structure of the heat kernel in Theorem 3.1 and following the proof of [21, Prop. 3.28] that describes the mapping properties of the usual "full" b-calculus on polyhomogeneous functions. ∎

We now discuss trace expansions of the heat kernel on the diagonal. To start, we review residue densities; cf. [26]. Let $Q \in \Psi_b^m(X)$ where $m \in \mathbb{Z}$. Let $\mathcal{U}_y \times \mathbb{R}^n_z$ be a coordinate patch on X_b^2 overlapping Δ_b such that $\Delta_b \cong \mathcal{U} \times \{0\}$. Then on this coordinate patch, we can write (see Remark 2.3)

$$K_Q = \int e^{iz\cdot\xi} q(y,\xi) d\!\!\!{}^-\xi \cdot d\mathfrak{m}',$$

where $q(y,\xi)$ is a classical symbol of order m. Then the Wodzicki residue density of Q is by definition

$$\operatorname{Res}(Q) = \int_{|\xi|=1} q_{-n}(y,\xi) d\!\!\!{}^-\xi \cdot d\mathfrak{m}(y), \tag{3.9}$$

where $q_{-n}(y,\xi)$ is the homogeneous component of $q(y,\xi)$ of degree $-n$. It is a remarkable property that $\operatorname{Res}(Q)$ is defined independent of coordinates; see [17]. Thus, the local definitions (3.9) produce a global density $\operatorname{Res}(Q) \in C^\infty(X, \Omega_b)$, where Ω_b is the b-density bundle (i.e., the span of the b-measure $d\mathfrak{m}$).

Now let $Q(\lambda) \in x^{-\beta}\Psi^{m,d}_{c,\Lambda}(X)$ be holomorphically tempered. Then $Q(\lambda)$ is holomorphic on a neighborhood of Λ by definition. Hence, we can define the Mellin transform of $Q(\lambda)$ to be the operator

$$\mathcal{M}(Q)(z) = \frac{i}{2\pi}\int_\Gamma \lambda^z Q(\lambda) d\lambda,$$

where Γ is the contour (3.2) with the number a in (3.2) now chosen to be a (small) positive number, and where λ^z is defined by its standard branch.

By computations similar to those in Theorem 3.1 and by using the results of [20, Sec. 7], see especially Theorem 7.5 of loc. cit., one can prove that $\mathcal{M}(Q)(z) \in x^{-\beta-dz-d}\Psi_b^{zd+m+d}(X)$. In particular, for z of form $(k-m-n)/d-1$ where $k \in \mathbb{N}_0$, $\mathcal{M}(Q)(z)$ is of order $k-n$; so its residue density is defined, and

$$\operatorname{Res}(\mathcal{M}(Q)(\ell)) \in x^{-\beta+m+n-k}C^\infty(X,\Omega_b), \quad \ell = \frac{k-m-n}{d} - 1.$$

The operator $\partial_z\mathcal{M}(Q)(z)$ is an example of an operator with "log-polyhomogeneous" symbols as studied by Lesch [17]; these operators also have well-defined residue densities. Let $\mathcal{M}(Q)(z) = x^{-\beta-dz-d}Q(z)$, where $Q(z) \in \Psi_b^{zd+m+d}(X)$. Then it follows that $\partial_z\mathcal{M}(Q)(z) = -\log x^d\mathcal{M}(Q)(z) + x^{-\beta-dz-d}\partial_z Q(z)$. Thus, $\mathrm{Res}(\partial_z\mathcal{M}(Q)(\ell))$ for $\ell = (k-m-n)/d - 1$ where $k \in \mathbb{N}_0$, in general diverges logarithmically at the boundary. However, if $\ell \in \mathbb{N}_0$, it turns out that $\mathrm{Res}(\mathcal{M}(Q)(\ell)) = 0$ (this was used in [20, Lem. 6.9]). Thus, when $\ell \in \mathbb{N}_0$, it follows that

$$\mathrm{Res}(\partial_z\mathcal{M}(Q)(\ell)) \in x^{-\beta+m+n-k}C^\infty(X,\Omega_b), \quad \text{if } \ell \in \mathbb{N}_0.$$

The next theorem follows from Theorem 3.1, applying the results of [20, Lem. 6.16]. To avoid reproducing the arguments of loc. cit. we omit the details.

THEOREM 3.4: *Let $A \in x^{-m}\mathrm{Diff}_b^m(X)$, $m \in \mathbb{Z}^+$, be fully elliptic with respect to $\alpha \in \mathbb{R}$ on a sector Λ of the form (3.1). Then given $B \in x^{-\beta}\mathrm{Diff}_b^{m'}(X)$, let $Be^{-tA}|_\Delta$ denote the Schwartz kernel of Be^{-tA} restricted to the diagonal of X^2. Then as $t \to 0$, we have*

$$Be^{-tA}|_\Delta \sim \sum_{k=0}^{\infty} \gamma_k t^{(k-m'-n)/m}, \tag{3.10}$$

where $\gamma_k \in x^{-\beta+m'+n-k}C^\infty(X,\Omega_b)$. Moreover, if $\ell = (k-m'-n)/m$, then

$$\gamma_k = \frac{\Gamma(-\ell)}{m}\mathrm{Res}(BA^\ell) \text{ if } \ell \notin \mathbb{N}_0, \ \gamma_k = \frac{(-1)^{\ell+1}}{m\cdot \ell!}\mathrm{Res}(B\log AA^\ell) \text{ if } \ell \in \mathbb{N}_0,$$

where we define

$$\mathrm{Res}(BA^\ell) = \mathrm{Res}(\mathcal{M}(Q)(\ell)) \quad \text{and} \quad \mathrm{Res}(B\log AA^\ell) = \mathrm{Res}(\partial_z\mathcal{M}(Q)(\ell)), \tag{3.11}$$

with $Q(\lambda)$ the holomorphically tempered operator given in Theorem 2.4. The asymptotic sum (3.10) means that for any $N \in \mathbb{Z}^+$ we have

$$Be^{-tA}|_\Delta - \sum_{k=0}^{N-1} \gamma_k t^{(k-m'-n)/m} = t^{(N-m'-n)/m} r_N(t), \tag{3.12}$$

where $r_N(t)$ is bounded at $t=0$ with values in $x^{-\beta+m'+n-N}C^0(X,\Omega_b)$.

Note that the Schwartz kernel gets progressively singular at $x=0$ as N gets larger. Thus, the asymptotic sum (3.10) cannot be integrated over X to attain a trace expansion. The trace of Be^{-tA} will be examined in the next section.

3.2. TRACE EXPANSIONS. Our goal is to prove Theorem 1.1 of the introduction. We begin by reviewing the definition of the b-integral; cf. [22, Sec. 4.19].

LEMMA 3.5: *Let $u(x) \in x^{-\beta}C_c^\infty([0,1)_x)$ where $\beta \in \mathbb{R}$. Then for any nonzero $a \in \mathbb{C}$, the map $\mathbb{C} \ni z \mapsto \int x^{az}u(x)\frac{dx}{x}$ defines a meromorphic function on $\mathbb{C}$ with simple poles at $z = -(k-\beta)/a$, where $k \in \mathbb{N}_0$, with residue $1/(ak!)\partial_x^k(x^\beta u)(0)$.*

Proof: Expanding u in Taylor series gives $u \sim x^{-\beta}\sum_{k=0}^\infty x^k/k!\partial_x^k(x^\beta u)(0)$. Since $\int_0^1 x^{az-\beta+k}\frac{dx}{x} = 1/(az-\beta+k) = a^{-1}/(z+(k-\beta)/a)$ for $\mathrm{Re} z$ sufficiently large, our lemma follows. ∎

This lemma implies that given $u \in x^{-\beta}C^\infty(X,\Omega_b)$, where $\beta \in \mathbb{R}$, the map $\mathbb{C} \ni z \mapsto \int_X x^z u$ defines a meromorphic function on $\mathbb{C}$. The regular value of this map is called the b-integral of u and is denoted by ${}^b\!\int u$.

We are now ready to prove Theorem 1.1. Let $B \in x^{-\beta}\mathrm{Diff}_b^{m'}(X)$, $\beta \in \mathbb{R}$, $m' \in \mathbb{N}_0$, and let $A \in x^{-m}\mathrm{Diff}_b^m(X)$, $m \in \mathbb{Z}^+$ with $m > \beta$, be fully elliptic with respect to $\alpha \in \mathbb{R}$ on a sector Λ of the form (3.1).

Proof of Theorem 1.1: The structure of the Schwartz kernel of Be^{-tA} in Theorem 3.1 plus arguments similar to those in [22, Sec. 4.18] imply that Be^{-tA} is trace class on $x^{\alpha-m}L_b^2(X)$ for $t > 0$ with trace obtained by integrating the Schwartz kernel restricted to the diagonal.

Assume that B is supported away from Y. Then by Theorem 3.4, it follows that $\int_X \mathrm{tr}\, Be^{-tA}$ can be expanded as in (1.4) but without the second sum. Moreover, the same theorem implies the formula for a_k in (1.5), and that the constant term in the expansion is given by (1.7) but without the second term.

We now assume that $\mathrm{tr}\, Be^{-tA}$ is supported on a patch $[0,\varepsilon)_x \times \mathcal{U}_y$, where $\mathcal{U}_y$ is a coordinate patch on Y. We may assume that $d\mathfrak{m} = (dx/x)dy$ and that $\varrho|_{\Delta_b} = x$ on this patch. Now write $Be^{-tA} = Q(t) + T(t)$, where $Q(t)$ and $T(t)$ are given in (3.3). We analyze the trace of each of $Q(t)$ and $T(t)$.

Since $\mathrm{Tr}\, T(t)$ is the easiest to analyze, we start with $\mathrm{Tr}\, T(t)$. By Part (D) of Theorem 3.1, it follows that we can write

$$\mathrm{Tr}\, T(t) = \iint x^{-\beta} f(s, x/s, y)\frac{dx}{x}dy, \quad t = s^m,$$

where $f(s,v,y)$ is smooth in y; smooth in $s \in [0,\infty)$; and smooth in $v \in (0,\infty)$, vanishing to infinite order as $v \to \infty$, and can be expanded at $v = 0$ with the index set $\widetilde{E}(\alpha)$. Since $\mathrm{Re}\widetilde{E}(\alpha) > m$ (see the definition of $\widetilde{E}(\alpha)$ in (2.6)), the integral in x is convergent. Now changing variables $x \mapsto v = x/s$ gives

$$(3.13)\quad \mathrm{Tr}\, T(t) = s^{-\beta}\iint v^{-\beta} f(s,v,y)\frac{dv}{v}dy = t^{-\beta/m}\iint v^{-\beta} f(t^{1/m}, v, y)\frac{dv}{v}dy.$$

Since $f(s,v,y)$ is smooth at $s = 0$, it follows that $\operatorname{Tr} T(t)$ can be expanded as in (1.4) but with only the coefficients c_k. If $\beta = 0$, then by (3.13), the constant term in $\operatorname{Tr} T(t)$ is given by $\int_0^\infty \int_Y f(0,v,y) dy \frac{dv}{v}$; in particular, if $\hat{\zeta}_T(z) = \frac{1}{\Gamma(z)} \int_0^\infty v^z \int_Y f(0,v,y) dy \frac{dv}{v}$, then by standard facts on the Mellin transform (see [1, Ch. 4]), $\hat{\zeta}_T(z)$ is a meromorphic function on $\mathbb{C}$ such that

(3.14) $$\operatorname{Res}_0\{\Gamma(z)\hat{\zeta}_T(z)\} = \text{ constant term in } \operatorname{Tr} T(t) \text{ as } t \to 0,$$

where Res_0 signifies regular value. We will use this fact later.

Consider now $Q(t)$. Relying on the notation of Theorem 3.1, we can write

$$\operatorname{Tr} Q(t) = \iiint x^{-\beta} q(t/x^m, x, y, \xi) đ\xi dy \frac{dx}{x}.$$

Note that the variable y plays the role of a parameter. Thus, in what follows we omit the variable y for notational simplicity; at the end of this proof, we just need to remember to insert $\int dy$.

Let $M(z)$ be the Mellin transform of $\operatorname{Tr} Q(t)$. Then making the change of variables $t \mapsto tx^m$, we can write

$$M(z) = \int_0^\infty t^{z-1} \operatorname{Tr} Q(t) dt = \Gamma(z) \int x^{mz-\beta} q(z,x) \frac{dx}{x},$$

where

(3.15) $$q(z,x) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \int q(t,x,\xi) đ\xi dt.$$

By the Mellin inversion formula, we can write $\operatorname{Tr} Q(t)$ in terms of $M(z)$:

(3.16) $$\operatorname{Tr} Q(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z} M(z) dz,$$

where $c >> 0$. By Cauchy's theorem, the poles of $M(z)$ are responsible for the powers of t that occur in the expansion of $\operatorname{Tr} Q(t)$ as $t \to 0$; see [1, Ch. 4]. Thus, it remains to investigate the poles of $M(z)$.

In what follows, we denote $(j - m' - n)/m$ by z_j for $j \in \mathbb{N}_0$. Then by Lemma 6.16 and Lemma 7.11 of [20], it follows that $q(z,x)$ has simple poles whenever $z = -\ell$ where $\ell = z_j \notin \mathbb{N}_0$, with residue given by

(3.17) $$\operatorname{Res}_1 q(z,x)|_{z=-\ell} = \frac{1}{m} x^{\beta+m\ell} \operatorname{Res}(\mathcal{M}(Q)(\ell)), \quad \ell = z_j \notin \mathbb{N}_0.$$

Moreover, $q(-\ell, x) = 0$ for $\ell \in \mathbb{N}_0$ and $\ell \neq z_j$ for any j, and if $\ell = z_j \in \mathbb{N}_0$, then

(3.18) $$q(-\ell,x) = -\frac{1}{m} x^{\beta+m\ell} \operatorname{Res}(\partial_z \mathcal{M}(Q)(\ell)), \quad \ell = z_j \in \mathbb{N}_0.$$

Now, $\Gamma(z)$ has only simple poles occurring at $z = -\ell$ where $\ell \in \mathbb{N}_0$ with residue $(-1)^\ell/\ell!$. Thus, as $M(z) = \Gamma(z) \int x^{mz} x^{-\beta} q(z,x) \frac{dx}{x}$, by Lemma 3.5 and the meromorphic properties of $q(z,x)$ mentioned above, $M(z)$ has poles of order at most two occurring only when z is of the form $z = -\ell$, where for some $j \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$, we have $\ell = z_j = (k-\beta)/m \notin \mathbb{N}_0$ or $\ell = z_j = (k-\beta)/m \in \mathbb{N}_0$. The second order residues in each case are given by

$$\frac{\Gamma(-\ell)}{m \cdot k!} \partial_x^k \{ \frac{1}{m} x^{\beta + m\ell} \mathrm{Res}(\mathcal{M}(Q)(\ell)) \}|_{x=0}, \quad \ell \notin \mathbb{N}_0,$$

and

$$\frac{(-1)^\ell}{m \cdot k! \cdot \ell!} \partial_x^k \{ -\frac{1}{m} x^{\beta + m\ell} \mathrm{Res}(\partial_z \mathcal{M}(Q)(\ell)) \}|_{x=0}, \quad \ell \in \mathbb{N}_0.$$

Since $\beta + m\ell = k$, these second order residues of $M(z)$ are

$$\begin{aligned} &\frac{\Gamma(-\ell)}{m^2 k!} \partial_x^k \{ x^k \mathrm{Res}(\mathcal{M}(Q)(\ell)) \}|_{x=0}, \text{ if } \ell \notin \mathbb{N}_0; \\ -&\frac{(-1)^\ell}{m^2 \cdot k! \cdot \ell!} \partial_x^k \{ x^k \mathrm{Res}(\partial_z \mathcal{M}(Q)(\ell)) \}|_{x=0}, \text{ if } \ell \in \mathbb{N}_0. \end{aligned} \tag{3.19}$$

Since $\mathrm{Tr}\, Q(t)$ is given by the inverse Mellin transform of $M(z)$ (see (3.16)), by Cauchy's theorem and the meromorphic properties of $M(z)$ (which follow from the meromorphic properties of $\Gamma(z)$, $q(z,x)$, and Lemma 3.5), it follows that $\mathrm{Tr}\, Q(t)$ has an expansion of the form (1.4). Moreover (see [1, Ch. 4]), since the second order residues of $M(z)$ are given by (3.19), the log terms in $\mathrm{Tr}\, Q(t)$ are given exactly by (1.6) (after inserting $\int dy$). Also, the formula for a_k in (1.5) follows from the formulas (3.17) and (3.18).

We now examine the constant term of $\mathrm{Tr}\, Q(t)$ when $\beta = 0$. Indeed, since $\mathrm{Tr}\, Q(t)$ is the inverse Mellin transform of $M(z)$, the constant term is exactly the residue of $M(z)$ at $z = 0$. Since $\Gamma(z)$ has a simple pole at $z = 0$ with residue 1 and $q(z,x)$ is regular at $z = 0$ with value $-\frac{1}{m}\mathrm{Res}(\partial_z \mathcal{M}(Q)(0))$, and since $M(z) = \int x^{mz} \Gamma(z) q(z,x) \frac{dx}{x}$, by Lemma 3.5 the residue of $M(z)$ at $z = 0$ is

$$\mathrm{Res}_1 M(z)|_{z=0} = {}^b\!\!\int -\frac{1}{m} \mathrm{Res}(\partial_z \mathcal{M}(Q)(0)) + \frac{1}{m} \mathrm{Res}_0 \{ \Gamma(z) q(z,0) \}. \tag{3.20}$$

Let $v = x/s$ where $s = t^{1/m}$. Then observe that

$$\hat{\zeta}_Q(z) := \frac{1}{\Gamma(z)} \int_0^\infty v^z \, \mathrm{tr}\, Q(t)|_{v=x/s, x=0} \frac{dv}{v} = \frac{1}{\Gamma(z)} \int_0^\infty v^z \int q(v^{-m}, 0, \xi) đ\xi \frac{dv}{v},$$

where $\mathrm{tr}\, Q(t)|_{v=x/s, x=0}$ means first set $s = x/v$ in $\mathrm{tr}\, Q(t)$ and then set $x = 0$. By standard facts on the Mellin transform (see [1, Ch. 4]), the properties of $q(t,x,\xi)$

imply that $\hat{\zeta}_Q(z)$ is a meromorphic function on $\mathbb{C}$. In particular, $\hat{\zeta}_A(z;B) = \hat{\zeta}_Q(z) + \hat{\zeta}_T(z)$ is a meromorphic function on $\mathbb{C}$. Now, changing variables $v \mapsto v^{-1/m}$ yields

$$\Gamma(z)\hat{\zeta}_Q(z) = \frac{1}{m}\int_0^\infty v^{-z/m}\int q(v,0,\xi)đ\xi\frac{dv}{v} = \frac{1}{m}\Gamma(-z/m)q(-z/m,0),$$

where we used the definition of $q(z,x)$ in (3.15). Thus, $\frac{1}{m}\mathrm{Res}_0\{\Gamma(z)q(z,0)\} = \mathrm{Res}_0\{\Gamma(z)\hat{\zeta}_Q(z)\}$. This result, plus (3.20) combined with (3.14), and the fact that $\hat{\zeta}_A(z;B) = \hat{\zeta}_Q(z) + \hat{\zeta}_T(z)$, show that the constant term in $\mathrm{Tr}\, Be^{-tA}$ as $t \to 0$ is given exactly by (1.7). Our proof is now complete. ∎

4. Applications

In this last section, we present applications of Theorem 1.1 and the structure theorem of the heat kernel, Theorem 3.1. We begin with the zeta function. We now bring back the vector bundle E that we have been leaving out.

Let $A \in x^{-m}\mathrm{Diff}_b^m(X,E)$, $m \in \mathbb{Z}^+$, be fully elliptic with respect to $\alpha \in \mathbb{R}$ on a sector Λ of the form (3.1). Suppose that $(A-\lambda)^{-1}$ exists on a neighborhood of Λ. Then, in [18] we show that as a consequence of Theorem 3.1, the complex power A^z of A exists and defines an entire family of b-pseudodifferential operators satisfying $A^zA^w = A^{z+w}$ for $z,w \in \mathbb{C}$. Also by [18] it follows that given any $B \in x^{-\beta}\mathrm{Diff}_b^{m'}(X,E)$ where $\beta \in \mathbb{R}$ with $\beta < m$, for $\mathrm{Re} z < \min\{(-m'-n)/m, -\beta/m\}$, the operator BA^z is trace class on $x^{\alpha-m}L_b^2(X,E)$.

Proof of Theorem 1.2: Using the well-known formula for the complex powers in terms of the heat operator:

$$A^z = \frac{1}{\Gamma(-z)}\int_0^\infty t^{-z}e^{-tA}\frac{dt}{t}, \quad \mathrm{Re} z << 0,$$

we can write

$$\mathrm{Tr}\, BA^z = \frac{1}{\Gamma(-z)}\mathcal{M}(f)(-z),$$

where $\mathcal{M}(f)(z)$ is the Mellin transform of the function $f(t) = \mathrm{Tr}(Be^{-tA})$. This theorem now follows from the results on the poles of Mellin transforms found in [1, Sec. 4.3], using the expansion (1.4) of $\mathrm{Tr}(Be^{-tA})$ as $t \to 0$, plus the fact that $1/\Gamma(-z)$ vanishes for $z \in \mathbb{N}_0$. ∎

Assume now that A: $x^\alpha H_b^m(X,E) \longrightarrow x^{\alpha-m}L_b^2(X,E)$ is self-adjoint and positive. We prove Theorem 1.3 and Theorem 1.4. Please see the introduction for the notation used in the following proofs.

Proof of Theorem 1.3: The leading coefficient of the trace expansion (1.4) (with $B = \mathrm{Id}$) in Theorem 1.1 can be verified to be given by

$$\frac{\Gamma(n/m)}{m} {}^b\!\!\int_X \mathrm{Res}(A^{n/m}) = \frac{\Gamma(n/m)}{m(2\pi)^n} \int_{S^*X} \mathrm{tr}\{a(p,\omega)^{-n/m}\} dp d\omega,$$

where $a(p,\xi)$ is the principal symbol of A. The Karamata tauberian theorem (see [25, p. 122]) applied to the integral $\mathrm{Tr}\, e^{-tA} = \int_0^\infty e^{-t\lambda} dN(\lambda)$ now completes the proof. ∎

Proof of Theorem 1.4: Let $h(t,p)d\mathfrak{m}(p) = \mathrm{tr}\, e^{-tA}$ be the fiber trace of e^{-tA} above the point (p,p) on the diagonal. Then, $h(t,p) = \int_0^\infty e^{-t\lambda} d\, \mathrm{tr}\, e(p,p,\lambda)$. Substituting tx^m (where $x = x(p)$) for t, we find that

$$h(tx^m, p) = \int_0^\infty e^{-t\lambda} d\sigma(p,\lambda), \quad \sigma(p,\lambda) := \mathrm{tr}\, e(p,p,\lambda/x^m).$$

Substituting tx^m for t in Equation (3.12) of Theorem 3.4 gives

$$h(tx^m, p) d\mathfrak{m}(p) - t^{-n/m} c_A(p) d\mathfrak{m}(p) = t^{(1-n)/m} x^{1-n} r_1(t),$$

where $c_A(p)d\mathfrak{m}(p) = x^{-n}\Gamma(n/m)/m\, \mathrm{Res}(A^{-n/m})$, and where $r_1(t)$ is bounded at $t = 0$ with values in $x^{n-1}C^0(X, \Omega_b)$. It is straightforward to verify that

$$c_A(p)d\mathfrak{m}(p) = \frac{\Gamma(n/m)}{m(2\pi)^n} \int_{{}^bS^*_pX} \mathrm{tr}\{a_b(p,\omega)^{-n/m}\} d_b\omega \cdot d\mathfrak{m}(p),$$

where $a_b(p,\xi)$ is the b-principal symbol of $x^m A$. It follows that as $t \to 0$,

$$\int_0^\infty e^{-t\lambda} d\sigma(p,\lambda) = t^{-n/m} c_A(p) + \mathcal{O}\big(t^{-n/m+1/m}\big), \tag{4.1}$$

where the $\mathcal{O}$ estimate is uniform in the topology of $C^0(X)$. Applying the Karamata tauberian theorem now yields the spectral estimate (1.9) of Theorem 1.4. Note that since the $\mathcal{O}$ estimate on the right-hand side of (4.1) is uniform for $p \in X$, it follows from the proof of the Karamata tauberian theorem that the o estimate on the right-hand side of (1.9) is also uniform for $p \in X$. ∎

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